MATH 245 S24, Exam 1 Solutions

1. Carefully define the following terms: floor, tautology.
   Let $x \in \mathbb{R}$. Then integer n is the floor of x if it satisfies $n \le x < n+1$. We call proposition p a tautology if it is logically equivalent to T, i.e. if it is true in all cases.

2. Carefully state the following theorems: Simplification Theorem, De Morgan's Law (for Propositions).
   The Simplification Theorem says: For any propositions p, q, (a) if $p \wedge q$ is T, then p is T; or (b) $p \wedge q \vdash p$; or (c) $p \wedge q$ yields p. Any one of (a),(b), or (c) is correct. Some of you did more than one, which is fine. Some of you put them all in a blender, combining bits and pieces in some way. Depending on how much/little sense your definition smoothie made, you may or may not have lost points. De Morgan's Law says: For any propositions p, q, we have $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$ and also $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$.

3. Let $a, b \in \mathbb{N}_0$. Use the definition of $\le$ to prove that if $2a \le b$ then $2a^2 + ab \le b^2$.
   We will use a direct proof. Suppose that $2a \le b$. Then $b - 2a \in \mathbb{N}_0$. Also $b + a \in \mathbb{N}_0$, since $a, b \in \mathbb{N}_0$. But now the product $(b-2a)(b+a) \in \mathbb{N}_0$, i.e. $b^2 - ab - 2a^2 \in \mathbb{N}_0$. Hence $b^2 - (2a^2 + ab) \in \mathbb{N}_0$, so $2a^2 + ab \le b^2$.

4. Let $a, b \in \mathbb{Z}$. Suppose that $a|b$, and that a is even. Prove that b is even.
   Because $a|b$, there is some integer k with $ak = b$. Because a is even, there is some integer n with $a = 2n$. Substituting, we get $(2n)k = b$, i.e. $2(nk) = b$. Because nk is an integer (being the product of two integers), b is even.

5. Carefully state the Double Negation Theorem, and prove it without using truth tables.
   The Double Negation Theorem states: For any proposition p, we must have $p \equiv \neg\neg p$. To prove it, there are two cases. If p is T, then $\neg p$ is F, and $\neg\neg p$ is T. If instead p is F, then $\neg p$ is T, and $\neg\neg p$ is F. In both cases, p and $\neg\neg p$ have the same truth value.

6. Prove or disprove: For p, q arbitrary propositions, $p \vdash p \to q$.
   The statement is false. To disprove it, consider what happens with p is T and q is F. Here the hypothesis (p) is T, but the conclusion ($p \to q$) is F.

7. Let $x \in \mathbb{R}$. Prove that if x is not even, then $x + 2$ is not even.
   We use a contrapositive proof. Assume that $x + 2$ is even. Then $x + 2$ is an integer, and there is some integer n with $x + 2 = 2n$. We rewrite as $x = 2n - 2 = 2(n-1)$. Since $n - 1$ is an integer, x is even.

   NOTE: It is a trap to try a direct proof – "not even" does not mean "odd" for real numbers, even with Corollary 1.8 – think about $x = 0.5$.

8. Prove or disprove: For p, q, r arbitrary propositions, $(p \vee q) \wedge r \equiv p \vee (q \wedge r)$.
The statement is false. To disprove it, consider what happens when p is T and r is F (and q is T, but this one doesn't matter). Since r is F, $(p \vee q) \wedge r$ is F. Since p is T, $p \vee (q \wedge r)$ is T. The two propositions do not agree. Since they do not agree in this case, out of eight cases, they do not agree in all cases and are not logically equivalent.

NOTE: Some of you really wanted to use truth tables for this problem, even going so far as to beg me during the exam for permission to do so. That's like asking for permission to walk to LA instead of driving: yes you can do it, but it will take much longer and so many things can go wrong. For example, it is not enough to say that two columns "do not match" – you need to identify a specific row (i.e. one case out of eight) where the columns do not agree.

9. Simplify the following proposition as much as possible (with no quantifiers, inequalities, or compound propositions negated): $\neg\ \forall x\ \exists z\ \forall y,\ x < y \leq z$.
We begin by pulling the negation into the quantifiers: $\exists x\ \forall z\ \exists y,\ \neg(x < y \leq z)$.
We now recall what a double inequality means: $\exists x\ \forall z\ \exists y,\ \neg(x < y \wedge y \leq z)$.
We apply De Morgan's Law: $\exists x\ \forall z\ \exists y,\ (\neg(x < y)) \vee (\neg(y \leq z))$.
Lastly, we negate the inequalities: $\exists x\ \forall z\ \exists y,\ (x \geq y) \vee (y > z)$.
No further simplification is possible. This cannot be recombined into a double inequality!

10. Prove or disprove: $\forall x \in \mathbb{Z},\ |x-3| > 4 \to |x-4| > 3$.
The statement is true. We always begin by letting $x \in \mathbb{Z}$ be arbitrary.

DIRECT PROOF: Assume that $|x-3| > 4$. Now there are two cases:
Case $x - 3 > 4$: Here $x > 7$, so by subtracting 4 from both sides we get $x - 4 > 3$. Hence $|x-4| = x - 4 > 3$.
Case $x - 3 < -4$: Here $x < -1$, and so $x < 1$. Subtracting 4 from both sides we get $x - 4 < -3$. Hence $|x-4| = -(x-4) > -(-3) = 3$.
In both cases, $|x-4| > 3$.

ALTERNATE SOLUTION (submitted by a student): We prove the implication $|x-3| > 4 \to |x-4| > 3$ (call it $p \to q$ for compactness) in each of four cases:
Case $x \leq -2$: $|x-4| = -(x-4) = -x+4 \geq 6 > 3$, so $p \to q$ is trivially true.
Case $-1 \leq x \leq 3$: $|x-3| = -(x-3) = -x+3 \leq 4$, so $p \to q$ is vacuously true.
Case $3 < x \leq 7$: $|x-3| = x-3 \leq 4$, so again $p \to q$ is vacuously true.
Case $x \geq 8$: $|x-4| = x-4 \geq 4 > 3$, so $p \to q$ is trivially true.